Polyhedral Approaches for Stable Steiner Tree Problems

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Abstract

We present a study on models and methods for optimized Steiner tree network design under stability requirements. That is, for certain pairs of potential tree edges, at most one of the edges is allowed to be included in the final design. The introduced model generalizes a wide range of well-known combinatorial problems including the optimization of Steiner trees, spanning trees, shortest paths, and stables sets. In this work, we develop and compare effective exact solution methodologies based on integer programming. Polyhedral approaches are strengthened by both, cutting planes that are valid for known sub-polytopes and novel problem-specific classes of valid inequalities. Moreover, we devise constraint programming-based formulations using multi-commodity flow models. We present preliminary computational results on new problem-specific instances; and instances from the literature for two problem special cases based on spanning trees and rainbow-constrained Steiner trees. We show that our methods are capable of solving both unsolved existing problems and new stable Steiner trees with up to five thousand nodes.

Keywords : *Steiner tree, integer programming, network design, telecommunication.*

1 Introduction

One of the omnipresent requirements in strategic telecommunication network planning is to design a network that establishes simple connectivity among stakeholders. A cost-oriented model that is widely used in the literature and in practice is the Steiner tree problem (STP). The STP asks for an edge cost-minimal acyclic network that connects given terminal nodes to a distribution node. In contrast to spanning tree networks, it may include so-called Steiner nodes as intermediate points. We consider the discrete version, also called the Steiner tree problem in graphs, one of Karp's 21 famous NP-hard problems in combinatorial optimization ([4]).

In this work, we investigate the integration of stability requirements in Steiner tree design. We consider practical applications in which – for a given pair of potential links – at most one link can be implemented in the network, but not both. The motivation is essentially incompatibility, and we assume that corresponding conflicting pairs of links are known a priori. These additional requirements notably change the nature of the problem and consequently impact optimized networks. It is not evident how optimization techniques can be adapted to efficiently handle conflict requirements, and how stability affects optimal network costs.

In telecommunication network design, pairing Steiner trees with stability requirements facilitates important practical needs such as planarity, and incompatibilities that stem from contracts, regulations, or physical limitations. We illustrate corresponding challenges in Figure 1 which depicts a potential network infrastructure (left) and a practically feasible network in an urban setting[1]. In this example, 11 terminals (nodes 1-11) need to be connected to the distributor node (r), potentially using a subset of the 8 Steiner nodes (nodes 12-19). A conflict was identified between links $\{4, 15\}$ and $\{4, 16\}$ due to their very small physical distance. Similarly, at most one of the edges $\{6, 10\}$ ($\{10, 12\}$) and $\{17, 18\}$ ($\{17, 19\}$) can be selected in the network to avoid a complicating non-planar network layout. A cost-efficient solution avoiding edge conflicts that uses three Steiner nodes is shown in Figure 1 (right).

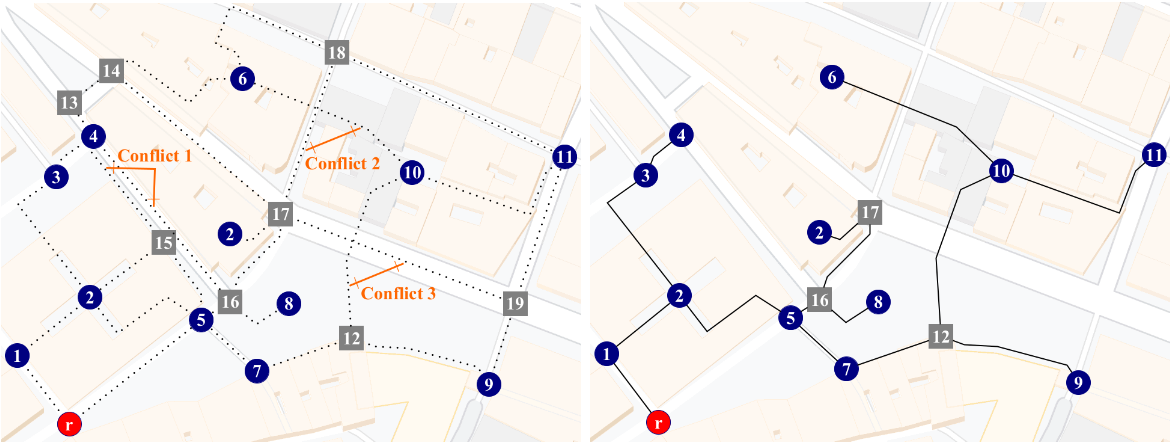

FIG. 1: An example for potential network infrastructure under three edge conflicts (left) and a feasible network connecting all 11 terminal nodes (1-11) using Steiner nodes 12, 16, and 17 (right) to the distributor r.

Further examples for relevant real-world network conflicts can for example be found in chip circuit design (VLSI), mine planning, and transportation planning. Related telecommunication data privacy concerns can give rise to unacceptable network configurations ([3]).

2 Optimization Methods

In this work, we develop formulations and algorithms for the novel problem using integer programming (IP). IP is known to be the state of the art for optimizing Steiner trees ([5]).

Let T be a set of terminal nodes, W a set of Steiner nodes, r a distributor node, and V the set of all nodes (i.e., $W \cup T \cup \{r\}$). We assume that T, W, and $\{r\}$ are disjoint, and denote the set of nodes excluding the root node by $V' = V \setminus \{r\}$. The set $E \subseteq 2^V$ contains all possible edges; the cost of an edge $e \in E$ is denoted by c_e. Now, let $C \subseteq 2^E$ be a set of edge conflicts. The stable Steiner tree problem (S-STP) asks for a solution such that at most one edge of each pair of conflict edges is part of a solution. We use binary edge variables y_e and arc variables x_a to mathematically formulate the S-STP in (F).

$$
\begin{aligned}
(F) \quad \min \quad & \sum_{e \in E} c_e y_e && (1)\\
\text{subject to:} \quad & x_{i,j} + x_{j,i} = y_{i,j} && (\{i,j\} \in E) \quad (2)\\
& y_{e_1} + y_{e_2} \leq 1 && (\{e_1, e_2\} \in C) \quad (3)\\
& \sum_{a \in \delta^-(S)} x_a \geq 1 && (S \cap T \neq \emptyset, S \subseteq V \setminus \{r\}) \quad (4)\\
& x_a \in \{0,1\} && (a \in A) \quad (5)\\
& y_e \in \{0,1\} && (e \in E) \quad (6)
\end{aligned}
$$

We integrate a broad range of valid inequalities into this non-compact arc-based formulation to tighten the polyhedral description. The main new cutting plane classes stem from:

[1] Map section showing the stock exchange square in Trieste (Italy), "Piazza della Borsa".

- Conflict-flow-balance inequalities
- Single and multi-star conflict-connectivity inequalities
- Symmetric and asymmetric clique inequalities
- Symmetric and asymmetric odd-cycle inequalities

We show how the exact separation of clique cuts is computationally feasible and superior to existing approximate approaches. Similarly, we present exact and approximate techniques to separate conflict-connectivity cuts. An exact method is developed that embeds cutting planes in a sophisticated branch-and-cut framework. We focus on both, strong dual bounds and time-effective primal solving capabilities. Additionally, we propose constraint programming (CP) formulations based on commodity flows to weigh out increasing number of variables against solver performance.

3 Computational Analysis

The S-STP can be used to model existing network optimization problems including the minimum spanning tree problem with conflicts (MST-C) ([1]), and the rainbow Steiner tree problem (R-STP) ([2]). In our preliminary experiments, we evaluate the presented optimization methods on three types of instances: a new set of SteinLib[2]-based instances that we generated; MST-C instances from the literature; R-STP instances from the literature. We carefully study the individual impact of our valid inequalities on the strength of formulation (F). It turns out that the are highly dependent on the problem instance class. Although selected cut classes can be separated efficiently, their combination may increase the overall computational effort significantly. Moreover, we investigate the capability of CP to generate good primal solutions for the S-STP. Using we are able to solve several MST-C and R-STP literature instances to optimality for the first time.

4 Conclusion and Perspectives

We present a new model to optimize Steiner trees under complicating stability requirements. Mathematical programming is used to compute primal and dual bounds ob optimal network costs. The development of novel cutting planes results in sugnificantly improved results; both optimality gaps and number of instances solved to optimality. In preliminary experiments we show that – even for problem sub-classes – our branch-and-cut methods are efficient and effective. Furthermore, we demonstrate how constraint programming models can be used to complement integer programming.

Our results can be used to better understand applicability and nature of Steiner trees when conflicts are present, and to provide guidance for practical implementation of efficient computational methods.

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